paths balance each other and so each provides the signal return path for the other. This design practice effectively eliminates RF currents that would occur on ground conductors. The off-chip RF interconnects interfacing the chip to the outside world also require differential signaling, but now, because of the larger electrical dimensions, differential transmission lines are required.

### 4.5 Modeling of Transmission Lines

Describing the signal on a line in terms of $E$ and $H$ requires a description of the $E$ and $H$ field distributions in the transverse plane. It is fortunate that current and voltage descriptions can be successfully used to describe the state of a circuit at a particular position along a TEM or quasi-TEM line. This is an approximation and the designer needs to be aware of situations where this breaks down. Such extraordinary effects are left to the next chapter. Once the problem of transmission line descriptions has been simplified to current and voltage, $R$, $L$, and $C$ models of a transmission line can be developed. A range of models are used for transmission lines depending on the accuracy required and the frequency of operation.

Uniform interconnects (with regular crosssection) can be modeled by determining the characteristics of the transmission line (e.g., $Z_0$ and $\gamma$ versus frequency) or arriving at a distributed lumped-element circuit, as shown in Figure 4-9. Typically EM modeling software models planar interconnects as having zero thickness, as shown in Figure 4-10. This is reasonable for microwave interconnects as the thickness of a planar strip is usually much less than the width of the interconnect. Many analytic formulas have also been derived for the characteristics of uniform interconnects. These formulas are important in arriving at synthesis formulas that can be used in design (i.e., arriving at the physical dimensions of an interconnect structure from its required electrical specifications). Just as importantly, they provide insight into the effects of materials and geometry.

Simplification of the geometry of the type illustrated in Figure 4-10 for microstrip can lead to appreciable errors in some situations. More elaborate computer programs that capture the true geometry must still simplify the real situation. An example is that it is not possible to account for density variations of the dielectric. Consequently characterization of many RF and microwave structures requires measurements to “calibrate” simulations.
Unfortunately it is also difficult to make measurements at microwave frequencies. Thus one of the paradigms in RF circuit engineering is to require measurements and simulations to develop self-consistent models of transmission lines and distributed elements.

### 4.6 Transmission Line Theory

Regardless of the actual structure, a segment of uniform transmission line (i.e., a transmission line with constant cross section along its length) can be modeled by the circuit shown in Figure 4-11(b). The primary constants can be defined as follows:

\[
\begin{align*}
\text{Resistance along the line} & = R \\
\text{Inductance along the line} & = L \\
\text{Conductance shunting the line} & = G \\
\text{Capacitance shunting the line} & = C
\end{align*}
\]

Thus \( R \), \( L \), \( G \), and \( C \) are also referred to as resistance, inductance, conductance, and capacitance per unit length. (Sometimes p.u.l. is used as shorthand for per unit length.) In the metric system we use ohms per meter (\( \Omega/m \)), henries per meter (\( H/m \)), siemens per meter (\( S/m \)) and farads per meter (\( F/m \)), respectively. The values of \( R \), \( L \), \( G \), and \( C \) are affected by the geometry of the transmission line and by the electrical properties of the dielectrics and conductors. \( G \) and \( C \) are almost entirely due to the properties of the dielectric and \( R \) is due to loss in the metal more than anything else. \( L \) is mostly a function of geometry, as most materials used with transmission lines have \( \mu_r = 1 \).

In most transmission lines the effects due to \( L \) and \( C \) tend to dominate because of the relatively low series resistance and shunt conductance. The propagation characteristics of the line are described by its loss-free, or lossless, equivalent line, although in practice some information about \( R \) or \( G \) is necessary to determine actual power losses. The lossless concept is just a useful and good approximation. The lossless approximation is not valid...
for narrow on-chip interconnections, as their resistance is very large.

### 4.6.1 Derivation of Transmission Line Properties

In this section the differential equations governing the propagation of signals on a transmission line are derived. Solution of the differential equations describes how signals propagate, and leads to the extraction of a few parameters that describe transmission line properties.

From **Kirchoff’s laws** applied to the model of Figure 4-11(b) and taking the limit as \( \Delta z \to 0 \) the transmission line or telegraphist’s equations are

\[
\begin{align*}
\frac{\partial v(z, t)}{\partial z} & = -R i(z, t) - L \frac{\partial i(z, t)}{\partial t} \quad (4.14) \\
\frac{\partial i(z, t)}{\partial z} & = -G v(z, t) - C \frac{\partial v(z, t)}{\partial t} . \quad (4.15)
\end{align*}
\]

For the sinusoidal steady-state condition with cosine-based phasors\(^{11}\) these become

\[
\begin{align*}
\frac{dV(z)}{dz} & = -(R + j\omega L)I(z) \quad (4.16) \\
\frac{dI(z)}{dz} & = -(G + j\omega C)V(z) . \quad (4.17)
\end{align*}
\]

Eliminating \( I(z) \) in Equations (4.16) and (4.17), yields the wave equation for \( V(z) \):

\[
\frac{d^2 V(z)}{dz^2} - \gamma^2 V(z) = 0 . \quad (4.18)
\]

Similarly,

\[
\frac{d^2 I(z)}{dz^2} - \gamma^2 I(z) = 0 , \quad (4.19)
\]

---

\(^{11}\) \( V(z) \) and \( I(z) \) are phasors and

\[
\begin{align*}
v(z, t) & = \Re \{ V(z)e^{j\omega t} \}, \quad i(z, t) = \Re \{ I(z)e^{j\omega t} \}. \\
\Re \{ w \} & \text{ denotes the real part of a complex number } w.
\end{align*}
\]
where the propagation constant is

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}.$$  \hfill (4.20)

In Equation (4.20) $\alpha$ is called the attenuation coefficient and has units of Nepers per meter; and $\beta$ is called the phase-change coefficient, or phase constant, and has units of radians per meter (expressed as rad/m or radians/m). Nepers and radians are dimensionless units, but serve as prompts for what is being referred to.

Equations (4.18) and (4.19) are second-order differential equations that have solutions of the form

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \quad \text{(4.21)}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}. \quad \text{(4.22)}$$

The physical interpretation of these solutions is that $V_0^+ e^{-\gamma z}$ and $I_0^+ e^{-\gamma z}$ are forward-traveling waves (moving in the $+z$ direction) and $V_0^- e^{\gamma z}$ and $I_0^- e^{\gamma z}$ are backward-traveling waves (moving in the $-z$ direction). Substituting Equation (4.21) in Equation (4.16) results in

$$I(z) = \gamma \frac{R + j\omega L}{R + j\omega L} [V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}]. \quad \text{(4.23)}$$

Then from Equations (4.23) and (4.22) we have

$$I_0^+ = \gamma \frac{R + j\omega L}{R + j\omega L} V_0^+; \quad I_0^- = \gamma \frac{R + j\omega L}{R + j\omega L} (-V_0^-). \quad \text{(4.24)}$$

Defining what is called the characteristic impedance as

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}. \quad \text{(4.25)}$$

Now Equations (4.21) and (4.22) can be rewritten as

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \quad \text{(4.26)}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}. \quad \text{(4.27)}$$

Converting back to the time domain:

$$v(z, t) = |V_0^+| \cos(\omega t - \beta z + \varphi^+) e^{-\alpha z}$$

$$\quad + |V_0^-| \cos(\omega t + \beta z + \varphi^-) e^{\alpha z}, \quad \text{(4.28)}$$

$$\text{where} \quad V_0^+ = |V_0^+| e^{j\varphi^+}, \quad \text{and} \quad V_0^- = |V_0^-| e^{j\varphi^-}. \quad \text{(4.30)}$$
and so the following quantities are defined:

- **Propagation constant:** \( \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \)  
  \( (4.31) \)
- **Attenuation constant:** \( \alpha = \Re\{\gamma\} \)  
  \( (4.32) \)
- **Phase constant:** \( \beta = \Im\{\gamma\} \)  
  \( (4.33) \)
- **Wavenumber:** \( k = -j\gamma \)  
  \( (4.34) \)
- **Phase velocity:** \( v_p = \frac{\omega}{\beta} \)  
  \( (4.35) \)
- **Wavelength:** \( \lambda = \frac{2\pi}{|\gamma|} = \frac{2\pi}{|k|} \)  
  \( (4.36) \)

where \( \omega = 2\pi f \) is the radian frequency and \( f \) is the frequency in hertz. The wavenumber \( k \) as defined here is used in electromagnetics and where wave propagation is concerned.\(^{12}\)

For low-loss materials (and for all of the substrate materials that are useful for transmission lines), \( \alpha \ll \beta \) and so \( \beta \approx |k| \), then the following approximates are valid:

- **Wavenumber:** \( k \approx \beta \)  
  \( (4.37) \)
- **Phase velocity:** \( v_p = \frac{\omega}{\beta} \)  
  \( (4.38) \)
- **Wavelength:** \( \lambda \approx \frac{2\pi}{|\beta|} = \frac{2\pi}{|k|} = v_p / f \).  
  \( (4.39) \)

The important result here is that a voltage wave (and a current wave) can be defined on a transmission line. One more parameter needs to be introduced: the group velocity,

\[ v_g = \frac{\partial \omega}{\partial \beta} . \]  
\( (4.40) \)

The group velocity is the velocity of a modulated waveform’s envelope and describes how fast information propagates. It is the velocity at which the energy (or information) in the waveform moves. Thus group velocity, can never be more than the speed of light in a vacuum, \( c \). Phase velocity, however, can be more than \( c \). For a lossless, dispersionless line, the group and phase velocity are the same. If the phase velocity is frequency independent, then \( \beta \) is linearly proportional to \( \omega \) and the group velocity is the same as the phase velocity \( (v_g = v_p) \).

**Electrical length** is often used in working with transmission line designs prior to establishing the physical length of a line. The electrical length of a transmission line is expressed either as a fraction of a wavelength or

---

\(^{12}\) There is an alternative definition for wavenumber, \( \nu = 1/\lambda \), which is used by physicists and engineers dealing with particles. When \( \nu \) is used as the wavenumber, \( k \) is referred to as the circular wavenumber or angular wavenumber.
in degrees (or radians), where a wavelength corresponds to 360° (or 2π radians). So if β is the phase constant of a signal on a transmission line and ℓ is its physical length, the electrical length of the line in radians is βℓ.

**EXAMPLE 4.2 Physical and Electrical Length**

A transmission line is 10 cm long and at the operating frequency the phase constant β is 30 m⁻¹ and the wavelength is 40 cm. What is the electrical length of the line?

**Solution:**

Let the physical length of the line be ℓ = 10 cm = 0.1 m. Then the electrical length of the line is ℓₑ = βℓ = (30 m⁻¹) × 0.1 m = 3 radians. The electrical length can also be expressed in terms of wavelength noting that 360° corresponds to 2π radians which corresponds to λ. Thus ℓₑ = (3 radians) = 3 × 180°/π = 171.9° or as ℓₑ = 3/(2π) λ = 0.477 λ.

**EXAMPLE 4.3 Forward- and Backward-Traveling Waves**

A transmission line ends (i.e., is terminated) in an open circuit. What is the relationship between the forward-traveling and backward-traveling voltage waves at the end of the line?

**Solution:**

At the end of the line the total current is zero, so that \( I^+ + I^- = 0 \) and so

\[
I^- = -I^+.
\]  (4.41)

Also, the forward-traveling voltage and forward-traveling current are related by the characteristic impedance:

\[
Z_0 = \frac{V^+}{I^+}.
\]  (4.42)

Similarly the backward-traveling voltage and backward-traveling current are related by the characteristic impedance:

\[
Z_0 = \frac{-V^-}{I^-},
\]  (4.43)

however, there is a change in sign, as there is a change in the direction of propagation. Combining Equations (4.41)–(4.43),

\[
Z_0 = \frac{V^+}{I^+} = -\frac{V^-}{I^-}
\]  (4.44)

and so substituting for \( I^- \),

\[
V^+ = -V^- I^+/I^- = -V^- I^-/(-I^+) = V^-- \]  (4.45)

So the total voltage at the end of the line, \( V_{\text{TOTAL}} \), is \( V^+ + V^- = 2V^+ \)—the total voltage at the end of the line is double the incident (forward-traveling) voltage.
EXAMPLE 4.4  

**RLGC Parameters**

A transmission line has the following RLGC parameters: $R = 100 \ \Omega m^{-1}$, $L = 80 \ \text{nH} \cdot \text{m}^{-1}$, $G = 1.6 \ \text{S} \cdot \text{m}^{-1}$, and $C = 200 \ \text{pF} \cdot \text{m}^{-1}$. Consider a traveling wave on the transmission line with a frequency of 2 GHz.

(a) What is the attenuation constant?
(b) What is the phase constant?
(c) What is the phase velocity?
(d) What is the characteristic impedance of the line?
(e) What is the group velocity?

**Solution:**

(a) $\alpha \gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$; $\omega = 12.57 \cdot 10^9 \ \text{rad} \cdot \text{s}^{-1}$

$$\gamma = \sqrt{(100 + j\omega \cdot 80 \cdot 10^{-9})(1.6 + j\omega \cdot 200 \cdot 10^{-12})} = 17.94 + j51.85 \ \text{m}^{-1}$$

$$\alpha = \Re\{\gamma\} = 17.94 \ \text{Np} \cdot \text{m}^{-1}$$

(b) Phase constant: $\beta = \Im\{\gamma\} = 51.85 \ \text{rad} \cdot \text{m}^{-1}$

(c) Phase velocity:

$$v_p = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \frac{12.57 \times 10^9 \ \text{rad} \cdot \text{s}^{-1}}{51.85 \ \text{rad} \cdot \text{m}^{-1}} = 2.42 \times 10^8 \ \text{m} \cdot \text{s}^{-1}$$

(d) $Z_0 = (R + j\omega L)/\gamma = (100 + j\omega \cdot 80 \cdot 10^{-9})/(17.94 + j51.85)$

$$Z_0 = 17.9 + j4.3 \ \Omega$$

Note also that

$$Z_0 = \sqrt{(R + j\omega L)/(G + j\omega C)}$$

which yields the same answer.

(e) Group velocity:

$$v_g = \frac{\partial \omega}{\partial \beta} \bigg|_{f=2 \ \text{GHz}}$$

Numerical derivatives will be used, thus

$$v_g = \frac{\Delta \omega}{\Delta \beta}.$$  

Now $\beta$ is already known at 2 GHz. At 1.9 GHz $\gamma = 17.884 + j49.397 \ \text{m}^{-1}$ and so $\beta = 49.397 \ \text{rad} \cdot \text{m}^{-1}$.

$$v_g = \frac{2\pi(2-1.9)}{51.85 - 49.397} = 2.563 \times 10^8 \ \text{m} / \text{s}.$$
4.6.2 Relationship to Signal Transmission in a Medium

In the previous section the telegraphists equations for a transmission modeled as subsections of $RLGC$ elements was derived. In this section these are related to signal transmission described by the physical parameters of permittivity and permeability. The development does not go into much detail as the derivation of the wave equations for a particular physical transmission line are involved and can only be derived for a few regular structures. If you are curious, the development is done for a parallel plate and rectangular waveguide in Appendix E on Page 871. The main parameters of describing propagation on a transmission line are $Z_0$ and $\gamma$, and these depend on the permeability and permittivity of the medium containing the EM fields, but also on the spatial variation of the $E$ and $H$ fields. As a result, $Z_0$ must be numerically calculated or derived analytically.

The propagation constant is derived from the field configurations as well with

$$\gamma^2 = -(k^2 - k_c^2)$$

(4.46)

where the wavenumber is

$$k = j\omega/\sqrt{\mu\varepsilon}$$

(4.47)

and $k_c$ is called the cutoff wavenumber. For TEM modes, $k_c = 0$. For non-TEM modes, $k_c$ requires detailed evaluation.

The other propagation parameters are unchanged:

$$\begin{align*}
\text{Attenuation constant:} & \quad \alpha = \Re\{\gamma\} \quad (4.48) \\
\text{Phase constant:} & \quad \beta = \Im\{\gamma\} \quad (4.49) \\
\text{Phase velocity:} & \quad v_p = \frac{\omega}{\beta} \quad (4.50) \\
\text{Wavelength:} & \quad \lambda = \frac{v_p}{f} \quad (4.51)
\end{align*}$$

where loss is incorporated in the imaginary parts of $\varepsilon$ and $\mu$. When $k_c = 0$ (as it is with coaxial lines, microstrip, and many other two-conductor transmission lines),

$$\gamma = j\omega/\sqrt{\mu\varepsilon}$$

(4.52)

Comparing $\gamma$ in Equation (4.20) and Equation (4.52), an equivalence can be developed between the lumped-element form of transmission line propagation and the propagation of an EM wave in a medium. Specifically,

$$-\omega^2\mu\varepsilon = (R + j\omega L)(G + j\omega C) \quad (4.53)$$

If the medium is lossless ($\mu$ and $\varepsilon$ are real and $R = 0 = G$), then

$$\mu\varepsilon = LC \quad (4.54)$$
When the medium is free space (a vacuum), then a subscript zero is generally used. Free space is also lossless, so the following results hold:

\[ \alpha_0 = 0 \quad \text{and} \quad \beta_0 = -j\gamma = \omega\sqrt{\mu_0\varepsilon_0}. \]  

(4.55)

If frequency is specified in gigahertz (indicated by \( f_{\text{GHz}} \))

\[ \beta_0 = 20.958 f_{\text{GHz}}. \]  

(4.56)

So at 1 GHz, \( \beta_0 = 20.958 \text{ rad} \cdot \text{m}^{-1} \). In a lossless medium with effective relative permeability \( \mu_r = 1 \) and effective relative permittivity \( \varepsilon_r \),

\[ \beta = \sqrt{\varepsilon_r} \beta_0. \]  

(4.57)

\( Z_0 \) depends strongly on the spatial variation of the fields. When there is no variation in the plane transverse to the direction of propagation

\[ Z_0 = \sqrt{\frac{\mu}{\varepsilon}}. \]  

(4.58)

However, if there is variation of the fields

\[ Z_0 = \kappa \sqrt{\frac{\mu}{\varepsilon}}, \]  

(4.59)

where \( \kappa \) captures the geometric variation of the fields.

If the boundary conditions on a transmission line are such that a required spatial variation of the fields cannot be supported then the signal cannot propagate. The critical frequency at which \( k = j\omega\sqrt{\mu\varepsilon} = k_c \) is called the cutoff frequency, \( f_c \). Signals cannot propagate on the line if the frequency is below \( f_c \).

### 4.6.3 Dimensions of \( \gamma, \alpha, \) and \( \beta \)

In the above expressions the propagation constant, \( \gamma \), is multiplied by length in determining impedance and signal levels. It is not surprising then that the SI units of \( \gamma \) are inverse meters (m\(^{-1}\)). The attenuation constant, \( \alpha \), and the phase constant, \( \beta \), have, strictly speaking, the same units. However, the convention is to introduce the dimensionless quantities Neper and radian to convey additional information. Thus the attenuation constant \( \alpha \) has the units of Nepers per meter (Np/m), and the phase constant, \( \beta \), has the units radians per meter (rad/m). The unit Neper comes from the name of the \( e (= 2.7182818284590452354 \ldots) \) symbol, which is called the Neper\(^{13} \) [65] or Napier’s constant. The number \( e \) is sometimes called Euler’s

---

\(^{13}\) The name is derived from John Napier, who developed the theory of logarithms described in his treatise *Mirifici Logarithmorum Canonis Descriptio*, 1614, translated as *A Description of the Admirable Table of Logarithms* (see [http://www.johnnapier.com/tableoflogarithms01.htm](http://www.johnnapier.com/tableoflogarithms01.htm)).
constant after the Swiss mathematician Leonhard Euler. The Neper is used in calculating transmission line signal levels, as in Equations (4.21) and (4.22). The attenuation and phase constants are often separated and then the attenuation constant, or more specifically \( e^{-\alpha} \), describes the decrease in signal amplitude per unit length as the signal travels down a transmission line. So when \( \alpha = 1 \) Np, the signal has decreased to \( \frac{1}{e} \) of its original value, and power drops to \( \frac{1}{e^2} \) of its original value. The decrease in signal level represents loss and, as with other forms of loss, it is common to describe this loss using the units of decibels per meter (dB/m). Thus 1 Np = \( 20 \log_{10} e \) = 8.685 dB. So expressing \( \alpha \) as 1 Np/m is the same as saying that the attenuation loss is 8.685 dB/m. To convert from dB to Np multiply by 0.1151. Thus \( \alpha = x \) dB/m = \( x \times 0.1151 \) Np/m. (Note that in engineering \( \log() \equiv \log_{10}() \) and \( \ln() \equiv \log_e() \).

**EXAMPLE 4.5  Transmission Line Characteristics**

A transmission line has an attenuation of 10 dB·m\(^{-1}\) and a phase constant of 50 radians·m\(^{-1}\) at 2 GHz.

(a) What is the complex propagation constant of the transmission line?

(b) If the capacitance of the line is 100 pF·m\(^{-1}\) and the conductive loss is zero (i.e., \( G = 0 \)), what is the characteristic impedance of the line?

**Solution:**

(a) \( \alpha|_{\text{Np}} = 0.1151 \times \alpha|_{\text{dB}} = 0.1151 \times (10 \text{ dB/m}) = 1.151 \text{ Np/m} \)

\( \beta = 50 \text{ rad/m} \)

Propagation constant, \( \gamma = \alpha + j\beta = 1.151 + j50 \text{ m}^{-1} \)

(b) \( \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \)

\( Z_0 = \frac{\sqrt{(R + j\omega L)(G + j\omega C)}}{G} \),

therefore \( Z_0 = \frac{\gamma}{G + j\omega C} \); \( \omega = 2\pi \times 2 \times 10^9 \text{ s}^{-1}; \ G = 0; \ C = 100 \times 10^{-12} \text{ F} \),

so \( Z_0 = 39.8 - 0.916 \Omega \).

4.6.4 *Lossless Transmission Line*

If the conductor and dielectric are ideal (i.e., lossless), then \( R = 0 = G \) and the equations for the transmission line characteristics simplify. The transmission line parameters from Equations (4.25) and (4.31)–(4.36) are then

\[ Z_0 = \sqrt{\frac{L}{C}} \]  
(4.60)

\[ \alpha = 0 \]  
(4.61)
\[ \beta = \omega \sqrt{LC} \]  \hspace{2cm} (4.62)

\[ v_p = \frac{1}{\sqrt{LC}} \]  \hspace{2cm} (4.63)

\[ \lambda_g = \frac{2\pi}{\omega \sqrt{LC}} = \frac{v_p}{f}. \]  \hspace{2cm} (4.64)

Note that there is a distinction between a transmission line and an RLC circuit. When referring to a transmission line having an impedance of 50 \( \Omega \), this is not the same as saying that the transmission line can be replaced by a 50 \( \Omega \) resistor. The 50 \( \Omega \) resistance is the characteristic impedance of the line. That is, the ratio of the forward-traveling voltage wave and the forward-traveling current wave is 50 \( \Omega \). It is not correct to call a lossless line reactive. Instead, the input impedance of a lossless line would be reactive if the line is terminated in a reactance. If the line is terminated in a resistance then the input impedance of the line would, in general, be complex, having a real part and a reactive part.

A transmission line cannot be replaced by a lumped element except as follows:

1. When calculating the forward voltage wave of a line which is infinitely long (or there are no reflections from the load). Then the line can be replaced by an impedance equal to the characteristic impedance of the line. The total voltage is then only the forward-traveling component.

2. The characteristic impedance and the load impedance can be plugged into the telegraphists equation (or transmission line equation) to calculate the input impedance of the terminated line.

### 4.6.5 Coaxial Line

The characteristic impedance of a transmission line is the ratio of the strength of the electric field to the strength of the magnetic field. The calculation of the impedance from the geometry of the line is not always possible except for a few regular geometries. For a coaxial line, the electric fields extend in a radial direction from the center conductor to the outer conductor. So it is possible to calculate the voltage by integrating this \( E \) field from the center to the outer conductor. The magnetic field is circular, centered on the center conductor, so the current on the conductor can be calculated as the closed integral of the magnetic field. Solving for the fields in the region between the center and outer conductors yields the following formula for the characteristic impedance of a coaxial line:

\[ Z_0 = \frac{138}{\sqrt{\varepsilon_r}} \log \frac{b}{a} \Omega, \]  \hspace{2cm} (4.65)

where \( \varepsilon_r \) is the relative permittivity of the medium between the center and outer conductors, \( b \) is the inner diameter of the outer conductor, and \( a \) is
Figure 4-12 Various coaxial transmission line adapters: (a) N-type female-to-female (N(f)-to-N(f)); (b) APC-7 to N-type male (APC-7-to-N(m)); (c) APC-7 to SMA-type male (APC-7-to-SMA(m)); SMA adapters: (d) SMA-type female-to-female (SMA(f)-to-SMA(f)); (e) SMA-type male-to-female (SMA(m)-to-SMA(f)); and (f) SMA-type male-to-male (SMA(m)-to-SMA(m)).

the outer diameter of the inner conductor. With a higher $\varepsilon$, more energy is stored in the electric field and the capacitance per unit length of the line $C$ increases. As the relative permittivity of the line increases, the characteristic impedance of the line reduces. Equation (4.65) is an exact formulation for the characteristic impedance of a coaxial line. Such a formula can only be approximated for nearly every other line.

Most coaxial cables have a $Z_0$ of 50 $\Omega$, but different ratios of $b$ and $a$ yield special properties of the coaxial line. When the ratio is 1.65, corresponding to an impedance of 30 $\Omega$, the line has maximum power-carrying capability. The ratio for maximum voltage breakdown is 2.7, corresponding to $Z_0 = 60 \, \Omega$. The characteristic impedance for minimum attenuation is 77 $\Omega$, with a diameter ratio of 3.6. A 50 $\Omega$ line is a reasonable compromise. Also the dimensions required for a 50 $\Omega$ line filled with polyethylene with a relative permittivity of 2.3 has dimensions that are most easily machined.

The velocity of propagation in a lossless coaxial line of uniform medium is the same as that for a plane wave in the medium. There is one caveat. This is true for all transmission line structures supporting the minimum variation of the fields corresponding to a TEM mode. Higher-order modes, with spatial variations of the fields, will be considered in Chapter 5. The diameter of the outer conductor and the type of internal supports for the internal conductor determine the frequency range of coaxial components. Various transmission line adapters are shown in Figure 4-12. It is necessary to convert between series and also to convert between the sexes (plug and jack) of connectors. The different construction of connectors can be seen more clearly in Figure 4-13. The APC-7 connector is shown in Figure 4-13(c). With this connector, the inside diameter of the outer conductor is 7 mm. The unique feature of this connector is that it is sexless with the interface plate being spring-loaded. These are precision connectors used in microwave measurements. The N-type connectors in Figures 4-13(a) and 4-13(b) are more common day-to-day connectors. There are a large number of
different types or series of connectors for high-power applications, different frequency ranges, low distortion, and low cost. There are also many types of coaxial cables, as shown in Figure 4-14(a). These are cables for use with SMA connectors (with 3.5 mm outer conductor diameter). These cables range in cost, flexibility, and the number of times they can be reliably flexed or bent. The semirigid cable shown at the bottom of Figure 4-14(a) must be bent using a bending tool, as shown in Figure 4-14(b) and in use in Figure 4-14(c). The controlled bending radius ensures minimal change in the characteristic impedance and propagation constant of the cable. Semirigid cables can only be bent once however. The highest precision bend is realized using an elbow bend, shown in Figure 4-14(d). Various flexible cables have different responses to bending, with higher precision (and more expensive) cables having the least impact on characteristic impedance and phase variations as cables are flexed. The highest-precision flexible cables are used in measurement systems.
EXAMPLE 4.6 Transmission Line Resonator

Communication filters are often constructed using several shorted transmission line resonators that are coupled to each other. Consider a coaxial line that is short-circuited at one end. The permittivity filling the coaxial line has a relative dielectric constant of 20 and the resonator is to be designed to resonate at a center frequency, \( f_0 \), of 1850 MHz when it is one-quarter wavelength long.

(a) What is the wavelength in the dielectric-filled coaxial line?

(b) What is the form of the equivalent circuit (in terms of inductors and capacitors) of the one-quarter wavelength long resonator if the coaxial line is lossless?

(c) What is the length of the resonator?

Solution: The first thing to realize with this example is that the first resonance will occur when the length of the resonator is one-quarter wavelength (\( \lambda/4 \)) long. Resonance generally means that the impedance is either an open or a short circuit and there is energy stored. When the shorted line is \( \lambda/4 \) long, the input impedance will be an open circuit and energy will be stored.

(a) \( \lambda_g = \frac{\lambda_0}{\sqrt{\varepsilon_r}} = \frac{16.2 \text{ cm}}{\sqrt{20}} = 3.62 \text{ cm} \).

(b) \( Y = Y_L + j\omega L + j\omega C \).

(c) \( \ell = \frac{(0.0362 \text{ m})}{4} = 9.05 \text{ mm} \).

4.6.6 Attenuation on a Low-Loss Line

Recall that \( \gamma \), the propagation constant, is given by

\[ \gamma = \sqrt{(R + j\omega L)(G + j\omega C)}. \]  

(4.66)

This can be written as

\[ \gamma = \omega \sqrt{LC} \left( 1 + \frac{R}{j\omega L} \right) \left( 1 + \frac{G}{j\omega C} \right). \]  

(4.67)

With a low-loss line, \( R \ll \omega L \) and \( G \ll \omega C \), and so using a Taylor series approximation

\[ \left( 1 + \frac{R}{j\omega L} \right)^{1/2} \approx 1 + \frac{1}{2} \frac{R}{j\omega L}. \]  

(4.68)
\[
\left(1 + \frac{G}{j\omega C}\right)^{1/2} \approx 1 + \frac{1}{2} \frac{G}{j\omega C},
\]  
(4.69)

thus

\[
\gamma \approx \frac{1}{2} \left( \frac{R}{\sqrt{L}} + \frac{G}{\sqrt{C}} \right) + j\omega \sqrt{LC}.
\]  
(4.70)

Hence for low-loss lines,

\[
\alpha \approx \frac{1}{2} \left( \frac{R}{Z_0} + GZ_0 \right),
\]  
(4.71)

\[
\beta \approx \omega \sqrt{LC}.
\]  
(4.72)

What Equation (4.72) indicates is that for low-loss lines the attenuation constant, \( \alpha \), consists of dielectric- and conductor-related parts; that is,

\[
\alpha = \alpha_d + \alpha_c,
\]  
(4.73)

where

\[
\alpha_d = GZ_0/2
\]  
(4.74)

is the loss contributed by the dielectric, called the dielectric loss, and

\[
\alpha_c = R/(2Z_0)
\]  
(4.75)

is the loss contributed by the conductor, called the ohmic or conductor loss.

For a microstrip line, an estimate of \( G \) is [51]

\[
G = \frac{\epsilon_e - 1}{\epsilon_r - 1} \omega \tan \delta \epsilon_r C_{\text{air}},
\]  
(4.76)

where \( \tan \delta \) is the loss tangent of the microstrip substrate. So from Equations (4.173) and (4.74)

\[
\alpha_d = \frac{GZ_0}{2} = \frac{1}{2} \frac{\epsilon_e - 1}{\epsilon_r - 1} \omega \tan \delta \epsilon_r C_{\text{air}} \frac{1}{\epsilon \sqrt{C_{\text{air}}}}.
\]  
(4.77)

Or, using Equation (4.176), this can be written as

\[
\alpha_d = \frac{\omega}{c} \tan \delta \epsilon_r \frac{\epsilon_r(\epsilon_r - 1)}{2\sqrt{\epsilon_r(\epsilon_r - 1)}} N_p \cdot \text{m}^{-1}.
\]  
(4.78)

### 4.6.7 Lossy Transmission Line Dispersion

On a lossy line, both phase velocity and attenuation constant are, in general, frequency dependent and so a lossy line is, in general, dispersive. That is, different frequency components of a signal travel at different speeds, and the phase velocity, \( v_p \), is a function of frequency. As a result the signal...
will spread out in time and, if the line is long enough, it will be difficult to extract the original information.

In the previous section it was seen, in Equation (4.72), that $\beta/\omega = v_p$ is approximately frequency independent for a low-loss line. Also, the conductor component of the attenuation constant, $\alpha_c$ in Equation (4.75), is approximately frequency independent. However, the dielectric component, $\alpha_d$ in Equation (4.78), is frequency dependent even for a low-loss line. If the transmission line has moderate loss, as with microstrip lines, all of the propagation parameters will be frequency dependent and the line is dispersive.

### 4.6.8 Design of a Dispersionless Lossy Line

The parameters that are important in describing the signal propagation properties of a transmission line are the propagation constant, $\gamma$, and the characteristic impedance, $Z_0$. Instead of $\gamma$ it is more appropriate to examine $\alpha$ and $v_p = \beta/\omega$ as these are the parameters that are ideally frequency independent if a signal, such as a pulse or modulated carrier, are to travel down the line and not be distorted. As was seen in the previous section these are generally frequency dependent for a lossy line. However, it is possible to design a line that is lossy but dispersionless, that is, $\alpha$, $\beta/\omega$, and $Z_0$ are independent of frequency. In this section a transmission line design is presented for a dispersionless line.

For any line the propagation constant is

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = j\omega \sqrt{LC} \left[ \left( 1 + \frac{R}{j\omega L} \right) \left( 1 + \frac{G}{j\omega C} \right) \right]^{1/2}. \tag{4.79}$$

If $R$, $L$, $C$, and $G$ are selected so that

$$\frac{R}{L} = \frac{G}{C}, \tag{4.80}$$

then for this case

$$\gamma = \alpha + j\beta = j\omega \sqrt{LC} \left( 1 + \frac{R}{j\omega L} \right) = R \sqrt{\frac{C}{L}} + j\omega \sqrt{LC}.$$

From this the attenuation constant, $\alpha$, and phase constant, $\beta$, are given by

$$\alpha = R \sqrt{\frac{C}{L}}, \quad \beta = \omega \sqrt{LC}, \tag{4.81}$$

and the phase velocity is

$$v_p = \frac{1}{\sqrt{LC}}. \tag{4.82}$$
To complete the analysis consider the characteristic impedance

\[ Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{L}{C}} \sqrt{\frac{R/L + j\omega}{G/C + j\omega}} \]  

(4.83)

and referring to Equation (4.80) it is seen that the second square root is just 1, so

\[ Z_0 = \sqrt{\frac{L}{C}}, \]  

(4.84)

which is frequency independent. So the important characteristics describing signal propagation are independent of frequency and so the transmission line is dispersionless.

4.7 The Terminated Lossless Line

4.7.1 Total Voltage and Current

Consider the terminated line shown in Figure 4-15. Assume an incident or forward-traveling wave, with traveling voltage \( V_0^+ e^{-j\beta z} \) and current \( I_0^+ e^{-j\beta z} \), respectively, propagating toward the load \( Z_L \) at \( z = 0 \). The characteristic impedance of the transmission line is the ratio of the voltage and current traveling waves so that

\[ \frac{V_0^+ e^{-j\beta z}}{I_0^+ e^{-j\beta z}} = \frac{V_0^+}{I_0^+} = Z_0. \]  

(4.85)

The reflected wave has a similar relationship (but watch the sign change):

\[ \frac{V_0^- e^{j\beta z}}{-I_0^- e^{j\beta z}} = \frac{V_0^-}{-I_0^-} = Z_0. \]  

(4.86)

The load \( Z_L \) imposes an additional constraint on the relationship of the total voltage and current at \( z = 0 \):

\[ \frac{V_L}{I_L} = \frac{V(z = 0)}{I(z = 0)} = Z_L. \]  

(4.87)