$$\Gamma_{in} = A_{11}^2 + \frac{A_{12}^2 A_{21}^2 T_{01}^2 B \Gamma_{11}}{1 - T_{01}^2 B \Gamma_{11} A T_{22}^2}$$

$$= 0.333 + \frac{0.943 \times 0.943 \times (0.707 - j0.707)^2 \times (-0.2)}{1 - (0.707 - j0.707)^2 \times (-0.2) \times (-0.333)}$$

$$= 0.345 - j0.177 \right) \quad (4.149)$$

$\Gamma_{in}$ is the reflection at reference plane A and is referenced to $Z_1 = 30 \, \Omega$. So the input impedance is

$$Z_{in} = Z_1 \left( \frac{1 + \Gamma_{in}}{1 - \Gamma_{in}} \right) = 30 \left( \frac{1 + 0.345 - j0.177}{1 - 0.345 - j0.177} \right) = 55.39 + j23.08 \, \Omega. \quad (4.150)$$

(f) Use the lossless telegraphist’s equation, Equation (4.128), to find $Z_{in}$. The infinitely long line presents an impedance $Z_{02}$ to the $60 \, \Omega$ transmission line. So the input impedance looking into the $60 \, \Omega$ line at reference plane A is, using the lossless telegraphist’s equation,

$$Z_{in} = Z_{01} \left( \frac{Z_{02} + jZ_{01} \tan \beta l}{Z_{01} + jZ_{02} \tan \beta l} \right),$$

where the electrical length $\beta l$ is $45^\circ$ or $\pi/8$ radians. So

$$Z_{in} = 60 \left( \frac{40 + j60 \tan (\pi/8)}{60 + j40 \tan (\pi/8)} \right) = 55.39 + j23.08 \, \Omega,$$

which is the same result obtained using the bounce diagram method.

In the above, two techniques were used to evaluate the input impedance of a transmission line system. The bounce diagram technique required a subtle choice of several different reference impedances which were required to keep track of real power flow. It is quite difficult to make these choices. It can be viewed that the bounce diagram considers instantaneous reflections of pulses. These are summed to yield the steady-state input reflection coefficient, and thus the input impedance. The telegraphist’s equation directly captures the steady-state response. So while the bounce diagram technique aids in physical understanding, using the telegraphist’s equation is a much less error prone approach to solving transmission line problems.

### 4.8 Special Cases of Lossless Terminated Lines

A number of special cases of lossless terminated transmission lines are useful circuit elements in RF design.
4.8.1 Electrically Short Lossless Line

Now consider the case of electrically short lines, that is, $\beta \ell$ goes to zero (i.e., $2\pi\ell/\lambda$ goes to zero), then Equation (4.128) becomes

$$Z_{\text{in}} \approx Z_L + jZ_0(\beta \ell) \left[1 \pm \frac{Z_L}{Z_0(\beta \ell)}\right].$$

This can be written as,

$$Z_{\text{in}} \approx Z_L \left[1 + (\beta \ell)^2\right] + j \left[\omega(L\ell) - Z_L^2\omega(C\ell)\right].$$

Thus the input impedance of an electrically short line terminated in impedance $Z_L$ is

$$Z_{\text{in}} \approx Z_L + j[\omega(L\ell) - Z_L^2\omega(C\ell)].$$

Some special cases of this result will be considered next.

4.8.2 Short Length of Short-Circuited Line

A transmission line terminated in a short circuit ($Z_L = 0$) has the input impedance

$$Z_{\text{in}} = jZ_0 \tan(\beta \ell).$$
So a short length of line, $\ell < \lambda_g/4$, looks like an inductor with inductance $L_s$, 

$$Z_0 \tan(\beta \ell) = \omega L_s, \tag{4.153}$$

and so

$$L_s = \frac{Z_0}{\omega} \tan \frac{2\pi \ell}{\lambda_g}. \tag{4.154}$$

From Equation (4.154) it is seen that for a given $\ell$, $L_s$ is proportional to $Z_0$. Hence, for larger values of $L_s$, sections of transmission line of high characteristic impedance are needed. So microstrip lines with narrow strips can be used to realize inductors in planar microstrip circuits.

### EXAMPLE 4.17 Inductive Transmission Line

This example demonstrates that one does not need an ideal short to effect a (predominantly) inductive behavior from a segment of transmission line. Consider the transmission line system shown below with lines having two different characteristic impedances, $Z_{01}$ and $Z_{02}$, $Z_{02} \ll Z_{01}$.

The value of $Z_{in}$ is

$$Z_{in} = Z_{01} \frac{Z_{02} + jZ_{01} \tan \beta \ell}{Z_{01} + jZ_{02} \tan \beta \ell},$$

which for a short line can be expressed as

$$Z_{in} \approx Z_{02} [1 + \tan(\beta \ell)] + jZ_{01} \tan(\beta \ell).$$

Note that $jZ_{01} \tan(\beta \ell)$ is the dominant part for $l < \lambda/8$ and $Z_{02} \ll Z_{01}$.

From this example it can be concluded that a microstrip realization of a series inductor can be a high-impedance line embedded between two low-impedance lines. A top view of such a configuration in microstrip is shown in the figure below. Recall that a narrow microstrip line has high characteristic impedance.
4.8.3 Short Length of Open-Circuited Line

An open-circuited line has $Z_L = \infty$ and so

$$Z_{\text{in}} = -j \frac{Z_0}{\tan \beta \ell}.$$  \hspace{1cm} (4.155)

For lengths $\ell$ such that $\ell < \lambda/4$, an open-circuited segment of line realizes a capacitor $C_0$ for which

$$\frac{1}{\omega C_0} = \frac{Z_0}{\tan \beta \ell}$$  \hspace{1cm} (4.156)

and so

$$C_0 = \frac{1}{Z_0} \frac{\tan \beta \ell}{\omega}.$$  \hspace{1cm} (4.157)

From the above relationship, we see that for a given length of line $\ell$, $C_0$ is inversely proportional to $Z_0$. Hence for larger values of $C_0$ we need to use sections of transmission line of low characteristic impedance.

**EXAMPLE 4.18 Transmission Line Filter**

A microstrip realization of a lowpass filter consists of $L$s and $C$s as shown in Figure 4-23(a). This lowpass filter section can be realized using wide and narrow microstrip lines, as shown in Figure 4-23(b).

4.8.4 Quarter-Wave Transformer

The next important case of a transmission line section is the quarter-wave transformer. Figure 4-24(a) shows a resistive load $R_L$ and a section of transmission line whose length $\ell$ is $\lambda_g/4$ (hence the name quarter-wave transformer). The input impedance of the line is

$$Z_{\text{in}} = Z_1 \frac{R_L + jZ_1 \tan(\beta \ell)}{Z_1 + jR_L \tan(\beta \ell)} = Z_1 \frac{R_L + jZ_1 \infty}{Z_1 + jR_L \infty} = \frac{Z_1^2}{R_L}.$$  \hspace{1cm} (4.158)
So the input impedance is matched to the transmission line $Z_0$ if

$$Z_{\text{in}} = Z_0;$$  \hfill (4.159)

that is,

$$Z_1 = \sqrt{Z_0 R_L}. \hfill (4.160)$$

Thus the quarter-wavelength long line acts as an ideal impedance transformer.

Another example of the quarter-wave transformer is shown in Figure 4-24(b). It is clear from the figure that the input impedance looking into the quarter-wave transformer is given by

$$Z_{\text{in}} = Z_0 \frac{Z_{01} + jZ_0 \tan(\beta \ell)}{Z_0 + jZ_{01} \tan(\beta \ell)} = \frac{Z_{01} + jZ_0 \infty}{Z_0 + jZ_{01} \infty} = \frac{Z^2_0}{Z_{01}}. \hfill (4.161)$$

Hence a section of the transmission line of length $\ell = \frac{\lambda_0}{4} + n\frac{\lambda_0}{2}$, where $n = 0, 1, 2, \ldots$ can be used to match two different impedances $Z_{01}, Z_{02}$ by constructing the line such that its characteristic impedance $Z_0$ is

$$Z_0 = \sqrt{Z_{01} Z_{02}}. \hfill (4.162)$$

Note that for a design center frequency $f_0$, the electrical length of the matching section is $\frac{\lambda_0}{4}$, but at different frequencies the electrical length is different and a perfect match is no longer achieved. In general, a perfect match is obtained only at the frequencies at which $\ell = \frac{\lambda_0}{4} + n\frac{\lambda_0}{2}$.

There is an interesting property of a quarter-wave transformer that is widely used in filters. Examine the final result in Equation (4.161) which is repeated here:

$$Z_{\text{in}} = \frac{Z^2_0}{Z_{01}}. \hfill (4.163)$$

This result applies to complex impedances as well. Equation (4.163) indicates that a quarter-wavelength long line is an impedance inverter, presenting, at Port 1 the inverse of the impedance presented at Port 2, $Z_{01}$. This inversion is normalized by the square of the characteristic impedance.
4.9 Input Impedance of a Terminated Lossy Line

In this section a lossy transmission line with low-loss is considered so that \( R \ll \omega L \) and \( G \ll \omega C \), and the characteristic impedance is \( Z_0 \approx \sqrt{L/C} \). Figure 4-26 is a lossy transmission line whose attenuation constant is \( \alpha \) and the phase constant is \( \beta \). The propagation constant, \( \gamma \), is a complex function of both \( \alpha \) and \( \beta \). The total voltage and current at any point on the line are given by

\[
V(z) = V_0^+ \left[ e^{-\gamma z} + \Gamma e^{\gamma z} \right]
\]

and

\[
I(z) = \frac{V_0^+}{Z_0} \left[ e^{-\gamma z} - \Gamma e^{\gamma z} \right].
\]

In a similar fashion, the impedance seen looking into the line for this case is given by the lossy telegraphist’s equation:

\[
Z_{in} = Z_0 \frac{Z_L + Z_0 \tanh \gamma \ell}{Z_0 + Z_L \tanh \gamma \ell}.
\]
For a lossless line this becomes

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta \ell}{Z_0 + jZ_L \tan \beta \ell} \quad (4.167)$$

as $\gamma = \alpha + j\beta$ and with $\alpha = 0$, $\tanh \gamma \ell$ becomes $\tanh j\beta \ell = j \tan \beta \ell$.

For a lossy transmission line it is clear that not all of the power applied at the input will be delivered to the load. There will be a certain amount of power lost on the line due to the finite attenuation constant that results from loss on the line. Equations (4.168)–(4.170) are the equations of power delivered to the load, the power incident on the line, and the power lost on the line, respectively. So the power delivered to the load is

$$P_L = \frac{1}{2} \Re \{ V(0)I^*(0) \} = \frac{|V_0^+|^2}{2Z_0} (1 - |\Gamma|^2) . \quad (4.168)$$

Similarly the power at the input of the line is

$$P_{in} = \frac{1}{2} \Re \{ V(-\ell)I^*(-\ell) \} = \frac{|V_0^+|^2}{2Z_0} \left[ 1 - |\Gamma|^2 e^{-4\alpha \ell} \right] e^{2\alpha \ell} . \quad (4.169)$$

and the power lost in the line is

$$P_{loss} = P_{in} - P_L = \frac{|V_0^+|^2}{2Z_0} \left[ (e^{2\alpha \ell} - 1) + |\Gamma|^2 (1 - e^{-2\alpha \ell}) \right] . \quad (4.170)$$

### 4.10 Microstrip Transmission Lines

Transmission lines with conductors embedded in an inhomogeneous dielectric medium cannot support a pure TEM mode. This is the case even if the conductors are lossless. The most important member of this class is the microstrip transmission line (Figure 4-8(c)). Part of the field is in the air and part in the dielectric between the strip conductor and ground. In most practical cases, the dielectric substrate is electrically thin, that is, $h \ll \lambda$. It can be shown that the transverse field is dominant, $|E_L| \ll |E_T|$, and the fields are called quasi-TEM.

#### 4.10.1 Microstrip Line in the Quasi-TEM Approximation

In this section it is shown that the static solutions for the transverse electric field alone can be used to calculate the characteristics of a transmission line. The procedure described is used in many EM computer programs to calculate the characteristics of transmission lines.

As a first step, the potential of the conductor strip is set to $V_0$ and Laplace’s equation is solved for the electrostatic potential everywhere in the dielectric. Then, the p.u.l. electric charge, $Q$, on the conductor is determined. Using this